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THE DETERMINISTIC MODEL WITH TIME DELAY FOR A NEW PRODUCT DIFFUSION IN A MARKET

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The aim of this paper is to present and analyze a market through a nonlinear deterministic model. A firm launches a new product and devotes a fixed proportion of sales to advertising, while customers go through a three stage adoption process with some delay on the effect of advertisement. The mathematical model is described by three nonlinear differential equations with time delay, where the word-of-mouth and advertising effectiveness are taken into account. The variables consist of the number of non-adopters (unaware of the existence of the product or the number of people who have not repurchased it), the number of thinkers (the number of people who know about the product, but they have not yet purchased it) and the number of adopters (the number of people who have purchased the product). The time delays are introduced in both purchase decisions of the thinkers and repurchase decisions of the adopters as well. The positive equilibrium point is determined and the conditions for the asymptotic stability are provided, when there is no delay. When the delay is taken as bifurcation parameter the conditions for the existence of a Hopf bifurcation are given. The critical value of the delay is found where the asymptotic stability is lost. Numerical simulations and conclusions can be found in the last part of the paper.

Keywords: **Consumers, Market, Advertising model, Equilibrium point, Stability**

JEL Classification: **C62, D11, E32, P46**

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1. Introduction

Different models related to the spread of a new product among potential customers are presented in the economic literature, by considering the effects of word-of-mouth, advertising, as well as other communication forms. The basic models can be found in (Bass, 1969), (Dodson, Muller, 1978) and (Lekvall, Wahlbin, 1973). Some of these models are described by nonlinear dynamical systems e.g. (Dhar, Tyagi, Sinha, 2010), (Feichtinger, Ghezzi, Piccardi, 1995), (Benito, González-Parra, Arenas, 2016), (Ciurdariu, Neamțu, 2012), (Sîrghi, Neamțu, 2013).

When a product is introduced into a market the role of advertising is important, because it could convince consumers to purchase the new product and therefore the sales increase and the profit is improved. Different types of factors such as economy, society, culture can limit the advertising effect.

The advertising diffusion model provided by (Feichtinger, Ghezzi, Piccardi, 1995), considers that a fixed proportion of sales is invested in advertising and the dynamics for the number of non-adopters (who do not purchase the product) and adopters (who purchase the product) is done. The model is described by two nonlinear differential equations, where the influence of word-of-mouth and advertising are introduced.

The advertising diffusion model from (Dhar, Tyagi, Sinha, 2010) improves the model from (Feichtinger, Ghezzi, Piccardi, 1995) by introducing one more variable in the economic process, more precisely the number of people who are exposed to advertising and will become adopters after a period of time (the thinkers). The dynamics of the nonlinear dynamical system with time delay is analyzed.

In (Sîrghi, Neamțu, 2013), in addition of (Dhar, Tyagi, Sinha, 2010), it is considered that the dynamics of the adopters class at moment t is affected by a proportion of the thinkers class at a previous moment, where the delay of the advertising effect is introduced. Also, the uncertainty about the environment is taken into account by the stochastic model.

We rely on (Dhar, Tyagi, Sinha, 2010), (Feichtinger, Ghezzi, Piccardi, 1995), (Benito, González-Parra, Arenas, 2016), (Sîrghi, Neamțu, 2013) to build a three compartment model where a firm launching a new product, a low priced, repurchased one and with three variables: the number of non-adopters (unaware of the existence of the product or the number of people who have not repurchased), the number of thinkers (who know about the product, but they have not yet purchased it) and the number of adopters (who have purchased the product). As in the previous models, the word-of-mouth and advertising effectiveness are used. Moreover, in this paper we consider that the adopters, who purchased the product in a previous moment t-τ, where τ is the time delay, can influence the thinkers at moment t in their decision to purchase the product and the adopters at moment t in their decision to repurchase it. The mathematical model is described by a system with three nonlinear delay differential equations with time delay.

The paper is organized as follows. In Section 2 the deterministic model of the market with three stages of a purchase decision is presented and the existence of the positive equilibrium points is established. The critical value of the delay parameter τ0 for which a Hopf bifurcation takes place is analyzed in Section 3. Section 4 provides the numerical simulations that verify the theoretical results. At the end, concluding remarks are given in Section 5.

2. The mathematical model

We present a model, where a firm launches a new low price product in a market with size population *N*, that can be repurchased. At moment $t \in R$, let $x(t)$ be the number of non-adopters, $y(t)$ be the number of thinkers and $z(t)$ be the number of adopters.

We suppose that with the rate a_1 the non-adopters become thinkers and with the rate a_3 the thinkers decide do not purchase the product, so that they will be non-adopters. The adopters who do not repurchased the product with the rate a_4 will be non-adopters. The interactions between non adopters and adopters are random and through word of mouth and lead to $\alpha x(t)z(t)$, where α is the word of mouth effectiveness. With respect to (Simon, Sebastian, 1987), $\alpha = a_2 z(t)$, where a_2 stands for the advertising effectiveness. Therefore, the number of non-adopters satisfies the following differential equation:

$$\dot{x}(t) = aN - (a_1 + a)x(t) + a_3 y(t) + a_4 z(t) - a_2 x(t) z(t)^2,$$

where *a* is the birth rate and represents the death rate.

The thinkers who will purchase the product will be adopters with the rate a_5. There is interaction between thinkers and adopters who purchased the product in a previous moment $t-\tau$ with the rate a_6 and between adopters at moment t and adopters at the previous moment $t-\tau$ with the rate a_7.

The number of thinkers and adopters satisfy the following differential equations:

$$\dot{y}(t) = a_1 x(t) - (a_5 + a_3 + a)y(t) + a_2 x(t)z(t)^2 - a_6 y(t)z(t-\tau) + a_7 z(t)z(t-\tau)$$

$$\dot{z}(t) = a_5 y(t) - (a_4 + a)z(t) + a_6 y(t)z(t-\tau) - a_7 z(t)z(t-\tau)$$

Thus, the deterministic model with time delay is given by the following nonlinear differential system:

$$\begin{aligned}
\dot{x}(t) &= aN - (a_1 + a)x(t) + a_3 y(t) + a_4 z(t) - a_2 x(t)z(t)^2 \\
\dot{y}(t) &= a_1 x(t) - (a_5 + a_3 + a)y(t) + a_2 x(t)z(t)^2 - a_6 y(t)z(t-\tau) + a_7 z(t)z(t-\tau) \\
\dot{z}(t) &= a_5 y(t) - (a_4 + a)z(t) + a_6 y(t)z(t-\tau) - a_7 z(t)z(t-\tau)
\end{aligned} \quad (1)$$

where $\tau \geq 0$, and $x(0) = x_0,\ y(0) = y_0,\ z(t) = f(t),\ f:[0,\tau] \rightarrow R,$ and $x(t) + y(t) + z(t) = N$ = constant.

3. The equilibrium point

The coordinates of the equilibrium point are the solutions of the following system:

$$\begin{aligned}
&aN - (a_1 + a)x + a_3 y + a_4 z - a_2 x z^2 = 0. \\
&a_1 x - (a_5 + a_3 + a)y + a_2 x z^2 - a_6 y z + a_7 z^2 = 0. \\
&a_5 y - (a_4 + a)z + a_6 y z - a_7 z^2 = 0.
\end{aligned} \quad (2)$$

From the third equation of (2), we have:

$$y = \frac{(a_4 + a)z + a_7 z^2}{a_5 + a_6 z}. \quad (3)$$

From $x(t)+y(t)+z(t)=N$ we obtain:

$$x=N-y-z=N-\frac{(a_4+a)z+a_7z^2}{a_5+a_6z}-z= \\ =N-\frac{(a_4+a+a_5)z+(a_6+a_7)z^2}{a_5+a_6z}. \qquad (4)$$

Replacing x from (5) and y from (3) in the first equation of (2) we get:

$$aN-(a_1+a++a_2z^2)(N-\frac{(a_4+a+a_5)z+(a_6+a_7)z^2}{a_5+a_6z})+a_4z+\frac{a_3((a_4+a)z+a_7z^2)}{a_5+a_6z}=0. \qquad (5)$$

From (5) it follows:

$$aN(a_5+a_6z)-(a_1+a++a_2z^2)(Na_5+Na_6-a_4-a-a_5)z-(a_6+a_7)z^2)+ \\ +a_4z(a_5+a_6z)+a_3(a_4+a)z+a_3a_7z^2=0 \qquad (6)$$

From (6) we obtain the equation:

$$A_4z^4+A_3z^3+A_2z^2+A_1z+A_0=0 \qquad (7)$$

where

$$\begin{aligned} A_4 &= a_2(a_6+a_7) \\ A_3 &= -a_2(Na_6-a_4-a-a_5) \\ A_2 &= (a_1+a)(a_6+a_7)-a_2Na_5+a_4a_6+a_3a_7 \\ A_1 &= aNa_6-(a_1+a)(Na_6-a_4-a-a_5)+a_4a_5+a_3(a_4+a) \\ A_0 &= -a_1a_5N. \end{aligned} \qquad (8)$$

Because $A_4 > 0$ and $A_0 < 0$ equation (7) has at least one positive solution. Therefore, we have:

Proposition 1:

If z_0 is a positive solution of (7), then the coordinates of the equilibrium E_0 are: x_0, y_0, z_0, where:

$$x_0 = N - \frac{(a_4 + a + a_5)z_0 + (a_6 + a_7)z_0^2}{a_5 + a_6 z_0},$$
$$y_0 = \frac{(a_4 + a)z_0 + a_7 z_0^2}{a_5 + a_6 z_0}. \quad (9)$$

4. The analysis of the equilibrium point E_0

In order to analyze the stability of the equilibrium point E_0, we study the linearized system of (1) in E_0, and we obtain:

$$\begin{aligned}
\dot{u}_1(t) &= a_{11}u_1(t) + a_{12}u_2(t) + a_{13}u_3(t)\\
\dot{u}_2(t) &= a_{21}u_1(t) + a_{22}u_2(t) + a_{23}u_3(t-\tau) + b_{23}u_3(t-\tau)\\
\dot{u}_3(t) &= a_{32}u_2(t) + a_{33}u_3(t-\tau) + b_{33}u_3(t-\tau)
\end{aligned} \quad (10)$$

where:

$$a_{11} = -a_1 a - a_2 z_0^2, \quad a_{12} = a_3, \quad a_{13} = a_4 - 2a_6 x_0 z_0,$$
$$a_{21} = a_1 + a_2 z_0^2, \quad a_{22} = -a_5 - a_3 - a - a_6 z_0, \quad a_{23} = 2a_2 x_0 z_0 + a_7 z_0, \quad b_{23} = a_6 y_0 + a_7 z_0,$$
$$a_{32} = a_5 + a_6 z_0, \quad a_{33} = -a_4 - a - a_7 z_0, \quad b_{33} = a_6 y_0 - a_7 z_0. \quad (11)$$

The characteristic equation of (10) is:

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0 \tag{12}$$

where:

$$\begin{aligned} &m_2 = -a_{11} - a_{22} - a_{33}, \quad m_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{32}a_{23}, \\ &m_0 = -a_{11}a_{22}a_{33} - a_{21}a_{32}a_{13} + a_{12}a_{21}a_{33} + a_{11}a_{32}a_{23}, \quad n_2 = -b_{33}, \\ &n_1 = -b_{33}(a_{11} + a_{33}) - a_{32}b_{23}, \quad n_0 = -a_{11}a_{22}b_{33} + a_{12}a_{21}b_{33} + a_{32}b_{23}a_{11}\ . \end{aligned} \tag{13}$$

Case 1: If $\tau = 0,$ the characteristic equation is:

$$\lambda^3 + m_{20}\lambda^2 + m_{10}\lambda + m_{00} = 0 \tag{14}$$

where:

$$m_{20} = m_2 + n_2, \quad m_{10} = m_1 + n_1, \quad m_{00} = m_0 + n_0. \tag{15}$$

Proposition 2:

If the conditions: $(H_1): m_{20} > 0, \ m_{20}m_{10} > m_{00}$ hold, all the roots of eq. (13) have negative real part. The positive equilibrium E_0 is locally asymptotically stable when $\tau = 0$ if the conditions (H_1) hold.

Case 2: $\tau > 0$

For $\tau > 0,$ let $\lambda = i\omega,$ $(\omega > 0),$ be the root of the eq. (13). Then we can obtain:

$$\begin{aligned} &M_{21}(\omega)\cos(\tau\omega) - M_{22}(\omega)\sin(\tau\omega) = M_{23}(\omega) \\ &M_{21}(\omega)\sin(\tau\omega) - M_{22}(\omega)\cos(\tau\omega) = M_{24}(\omega) \end{aligned} \tag{16}$$

where:

$$M_{21}(\omega) = n_1\omega, \quad M_{22}(\omega) = n_0 - n_2\omega^2,$$
$$M_{23}(\omega) = \omega^3 - m_1\omega, \; M_{24} = m_0\omega^2. \tag{17}$$

Then, we obtain:

$$\omega^6 + m_{22}\omega^4 + m_{21}\omega^2 + m_{20} = 0 \tag{18}$$

with:

$$m_{20} = -n^2, \quad m_{21} = m_1^2 - n_1^2 + 2n_0n_2, \quad m_{22} = m_2^2 - n_2^2 - 2m_1 \tag{19}$$

Denoting $\omega^2 = v_1$ then eq. (18) becomes:

$$v_1^3 + m_{22}v_1^2 + m_{21}v_1 + m_{20} = 0. \tag{20}$$

We define:

$$f(v_1) = v_1^3 + m_{22}v_1^2 + m_{21}v_1 + m_{20} \tag{21}$$

and we assume that the coefficients in $f(v_1)$ satisfy the condition:

$(H_2): a)\; m_{20} < 0,$ or $b)\; m_{20} \geq 0,\; \Delta_1 > 0,\; v_1^* > 0,$ and $f(v_1^*) = 0,$ where $\Delta_1 = m_{22}^2 - 3m_{21}$ and $v_1^* = \dfrac{-m_{22} + \sqrt{\Delta_1}}{3}$.

If the conditions (H_2) hold then eq. (20) has at least one positive root. We assume that eq. (20) has three positive roots which are denoted as v_{11}, v_{12}, v_{13}. Then eq. (18) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}$, $k = 1,2,3$. For every fixed ω_{1k}

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos \frac{g_{21}(\omega_{1k})}{h_{21}(\omega_{1k})} + \frac{2j\pi}{\omega_{1k}} \tag{22}$$

where:

$$\begin{aligned} g_{21}(\omega_{1k}) &= (n_1 - m_2 n_2)\omega_{1k}^4 + m_2 n_0 - m_1 n_1 \\ h_{21}(\omega_{1k}) &= (n_2\omega_{1k}^4 - n_0)^2 + n_1^2 \omega_{1k}^2, \quad k = 1,2,3, \quad j = 0,1,2. \end{aligned} \tag{23}$$

Define:

$$\tau_0 = \min\left\{\tau_{1k}^{(0)}\right\},\ \omega_{10} = \omega_{2k},\ k = 1,2,3. \tag{24}$$

Let $\lambda(\tau_0) = \alpha(\tau_0) + i\omega(\tau_0)$ be a root of eq. (12) near $\tau = \tau_0$. If $(H_2): f_1^{'}(v_{1k}) \neq 0,\ v_{1k} = \omega_0^2$ holds then $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} \neq 0$. Thus, we have:

Theorem 1 (Hassard, Kuznetov):

If the condition (H_2) holds, the positive equilibrium point $E_0(x_0, y_0, z_0)$ is locally asymptotically stable for $\tau \in [0, \tau_0)$. System (1) undergoes a Hopf bifurcation at $\tau = \tau_0$, and a branch of periodic solutions bifurcates from E_0 near $\tau = \tau_0$.

5. Numerical simulation

For the numerical simulation we use the following parameters: a =0.2, a_1 =0.6, a_2 =0.3, a_3 =0.5, a_4 =0.1, a_5=0.2, a_6 =0.8, a_7 =1, N =0.5. The equilibrium point is: x_0= 0.2348, y_0 =0.1556, z_0 =0.1093. The critical value of the bifurcation parameter is τ0=2.56. For any τ< τ0, the equilibrium point is locally asymptotically stable. For τ0=2.56 the solution of system (1) is periodically as we can visualize it in Figure 1.

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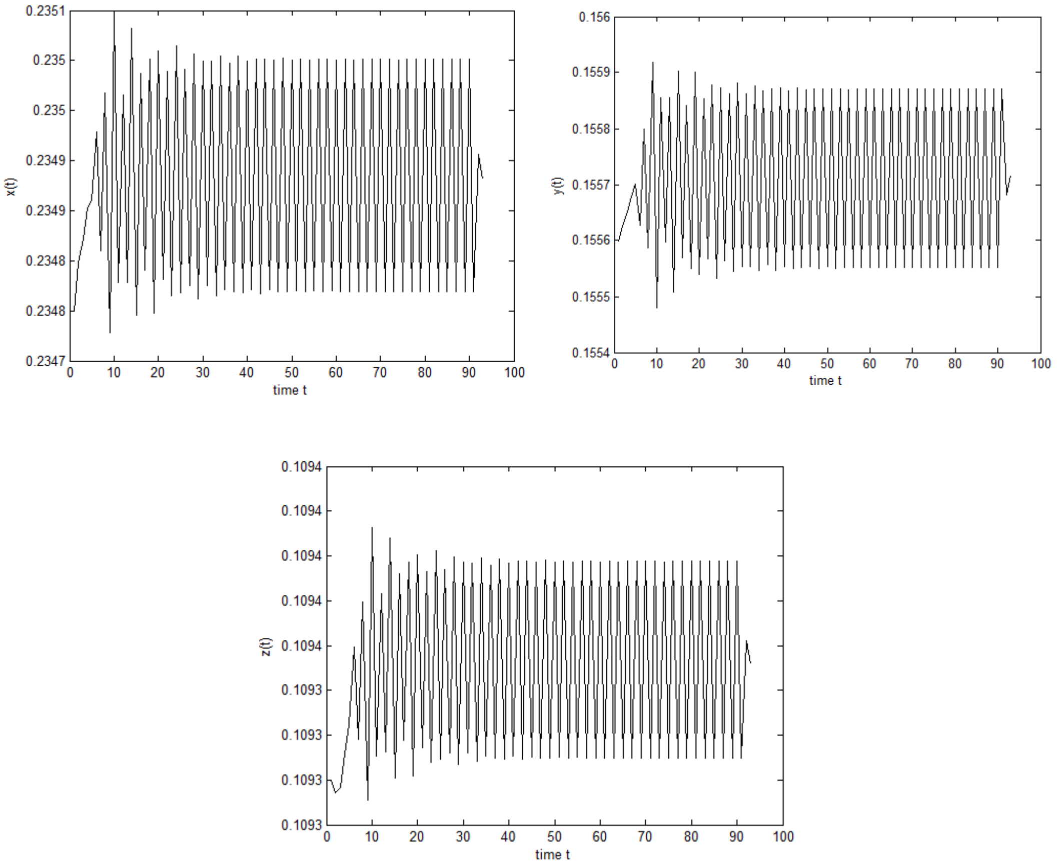

Figure 1. The orbits of system (1): (t, x(t)), (t, y(t)), (t, z(t)), for the critical value of the bifurcation parameter τ_{10}=2.56

Conclusion

In the present paper we present a model, where a firm launches a new low price product in a market. Three variables are considered: the number of non-adopters (unaware of the existence of the product or the number of people who have not repurchased), the number of thinkers (the number of people who know about the product, but they have not yet purchased it) and the number of adopters (the number of people who have purchased the product). In this process there is time delay in the purchase decision of the thinkers due to the interaction between them and adopters who purchased the product at the moment $t-\tau$ .Also, there is delay in the repurchase decision.

A system with three differential equations with time delay describes the deterministic model. We proved that there is at least one positive equilibrium point. The conditions for the local stability of the equilibrium point are determined. Considering the time delay τ as parameter, a Hopf-bifurcation takes place when $\tau = \tau_0$ and the system is locally stable when $\tau < \tau_0$ and it has an oscillatory character in $\tau = \tau_0$.

The numerical simulations are carried out using Maple and Matlab. They confirm the theoretical results.

Due to the fact that the environment, in which the system is operating, is rich in uncertainty in our future paper we will take into account the stochastic aspects as in (Mircea, Neamțu, Opriş, 2011).

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